Fine Pointing Control for a Large Spinning Spacecraft in Earth Orbit

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A large spinning spacecraft in Earth orbit includes a rigidly mounted telescope, parallel to the spacecraft's intended spin-axis. The three principal moments-of-inertia are unequal. Electric thrustors are used to overcome the gravity-gradient torque components. Uncertainty of the spacecraft's inertia tensor, including misalignment of the telescope axis from the actual principal direction, as well as magnetic, solar pressure, and aerodynamic forces, produce disturbing torques which are constant or vary sinusoidally during a spin-rotation. The constant torques and the coefficients of the sinusoidal torques are modeled as first-order Markov processes and consequently increase the dimension of the dynamical state. Measurements of the spacecraft's orientation are assumed to be of 2 types: a) the deviation, measured continuously in body axes, of the telescope axis from its desired direction; b) the deviation of the telescope axis towards or away from certain bright stars, measured as the star in question crosses a slit on the outside of the spacecraft. The estimation of the orientation from measurements a) and b) involves not only linear dynamical equations with periodic coefficients but also a mixture of a continuous and a discrete periodic, information pattern. The usual estimation gains settle here into a periodic pattern. Quadratic synthesis of a control law produces constant and periodically varying feedback gains. The constant control gains are obtained by eigenvector decomposition, and the periodic control gains by solving a linear matrix differential equation with periodic forcing term. The method of "spectral factorization" is used to obtain the estimator gains.

Introduction

A LARGE space telescope (LST)[1] is the next step for optical stellar space astronomy after the Orbital Astronomical Observatory (OAO) and the Apollo Telescope Mount (ATM). The location of a large telescope in space allows observations over the entire spectrum from about 1000Å to about 5u. This represents an increase of five times the available observation window from Earth. The atmospheric disturbances are removed and background brightness is much reduced. Thus a large telescope in space offers all the observational advantages to aid the advance of astronomical knowledge.

Spin stabilization of Pioneer spacecrafts has been very successful and more economical in fuel than 3-axis stabilization. We therefore look here at the problem of control of a Large Spinning Space Telescope (LSST). We assume that the LSST, in a near Earth circular orbit,(see Fig. 1), includes a rigidly-mounted telescope, parallel to the spacecraft's intended spin-axis which is, nominally, along a principal direction of inertia at the center of mass. Solar panels will supply the electrical power to the LSST. It will orbit Earth at an altitude of 800 km,[1] and it will spin at a spinrate of 5 rpm.

The three principal moments-of-inertia are unequal, so that the gravity-gradient torque fluctuates during a spin-rotation of the spacecraft. Body-fixed electric thrustors, mounted in pairs to provide torque components about the axes perpendicular to the telescope axis, are used to overcome the gravity-gradient torque components as well as the (smaller) torque components due to magnetic, aerodynamic, and solar-pressure forces.

Misalignment of the telescope axis from the actual principal direction produces unknown additional constant torque components. These constant torques are partly masked by slowly varying parts of the magnetic, aerodynamic, and solar-pressure torques, as well as slow fluctuations of the electric thrustors themselves about their intended output. The slow variation in the magnetic torques are due not only to change of the spacecraft's position relative to the Earth's magnetic field, but also to fluctuations, difficult to model, in the spacecraft's magnetic dipole moment due to onboard electrical activity. Fluctuations in the thrustor output are also necessarily unpredictable. The total "quasi-constant" torque components will therefore be modeled as first-order Markov random processes with time-constants large in comparison with the spin period. Small changes in the spacecraft's principal direction due to fuel expenditure and/or on-board mechanical activity will also be accounted for this way.

The gravity-gradient torque has fluctuating components perpendicular to the spin-axis, sinusoidal in the spin phase, and depending on the ratios of the principal moments-of-inertia as well as on position in orbit. The nominal gravity-gradient torque components will be balanced by nominal electric thrustor torques. Deviations from the nominal moment-of-inertia ratios will cause sinusoidal error torques which will be partially masked by sinusoidal components of the magnetic, aerodynamic and solar-pressure torques. Again, because of uncertainty about fluctuations in the spacecraft's dipole moment, the coefficients of the sine and cosine distur-

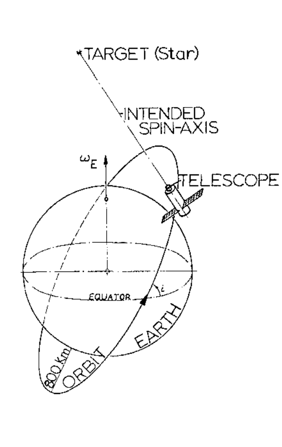

Fig. 1 Telescope in earth orbit.

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bing torques will be treated as first-order Markov random processes.

The orientation history of the spacecraft's telescope axis is now described by a 10th order linear system, the "state" consisting of 2 angles and their rates, 2 "constant" torques, and 4 coefficients of sinusoidal torques. Measurements of the spacecraft's orientation are assumed to be of 2 types: a) The deviation, measured continuously in body axes, of the telescope axis from its desired direction; b) the deviation of the telescope axis towards or away from certain bright stars, measured as the star in question crosses a slit on the outside of the spacecraft.

If the bright stars are not approximately uniformly distributed in direction around the desired direction, the measurements b) provide a discrete periodic, information pattern, rather than a constant continuous pattern as in a). The estimation of the 10-dimensional orientation state "x" from the measurements a) and b) involves not only linear dynamical equations with periodic coefficients but also a mixture of a continuous and a discrete, periodic, information pattern. The determination of the optimal state estimate $\hat{x}$ thus extends the work of Nishimura,[2] and of Breakwell, Kamel, Ratner.[3] The estimator gains settle into a periodic pattern (period = spin rate), obtainable by "spectral factorization."

The control may be synthesized by minimization of a suitable quadratic combination of the total telescope axis deviation and $(\delta u_1)^2+(\delta u_2)^2$, where δu_i $(i=1,2)$ are the additions to the nominal control which counteracts the nominal gravity-gradient torque. As in Ref. 3, this leads to a linear feedback law $[\delta u_1, \delta u_2]^T=-C(t)\,x$, where the matrix $C(t)$ is periodic with spin period. The noncontrollability of the additional 6 states describing the disturbance permits a partitioning of the matrix Riccati equation and a much simpler calculation of $C(t)$ than in Ref. 3.

Equations of Motion

The equations of motion of a rigid spinning body are

$$\overset{I}{H}=\overset{B}{H}+\omega^{B\text{-}I}\times H=T$$

where H = angular momentum of body, ω^{B-I} = angular velocity of body relative to inertial space, T = external torque, and superscripts I and B here denote time derivatives in inertial and body frames.

For a rigid body, H also can be written as $H=I.\omega^{B-I}$, where I is a diagonal matrix when the principal axes of inertial are chosen as the coordinate axes of the body. This substitution leads to the classical Euler equations.

$$I_x\dot{\omega}_x-(I_y-I_z)\omega_z\omega_y=T_x \tag{1a}$$

$$I_y\dot{\omega}_y-(I_z-I_x)\omega_z\omega_x=T_y \tag{1b}$$

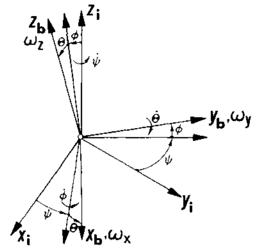
$$I_z\dot{\omega}_z-(I_x-I_y)\omega_x\omega_y=T_z \tag{1c}$$

Defining the Euler angles as in Fig. 2, for small angles ϕ, θ we get the relations

$$\dot{\phi}\simeq\omega_x+\omega_z\theta;\quad \dot{\theta}\simeq\omega_y-\omega_z\phi;\quad \dot{\psi}\simeq\omega_z=\text{constant} \tag{2}$$

Note that ϕ, θ are the (small) angular displacements of the body z-axis (z_b) towards x_b and $(-y_b)$.

Fig. 2 Euler angles.



Defining $K_x=(I_z-I_y/I_x)$, $K_y=(I_z-I_x/I_y)$, the equations of motion describing the orientation of the body axis z_b are

$$\dot{\omega}_x+K_x\dot{\psi}\,\omega_y\;=\;(T_x/I_x)=\bar{T}_x \tag{3a}$$

$$\dot{\omega}_y-K_y\dot{\psi}\,\omega_x\;=\;(T_y/I_y)=\bar{T}_y \tag{3b}$$

$$\dot{\phi}=\omega_x+\dot{\psi}\,\theta \tag{3c}$$

$$\dot{\theta}=\omega_y-\dot{\psi}\theta \tag{3d}$$

The disturbing torques acting on the spacecraft are: a) gravity- gradient torque T_{gg}, b) geomagnetic torque T_M, c) torque due to solar pressure T_s, d) aerodynamic torque T_{aer}, and e) torque due to micrometeorites T_m. In a near Earth orbit (800 km) the magnitudes of the first four disturbances have been estimated[4] to be in the ratios 1:0.25: 0.002: 0.0005. The torque due to micrometeorites is negligible.

Disturbances

Gravity-Gradient Torque

In a near-Earth orbit the gravity-gradient torque is the dominant disturbance acting on the spacecraft.

$$T_{gg}=3n^2\hat{r}\times I\cdot\hat{r} \tag{4}$$

where n ≙ orbital rate, and $\hat{r}$ ≙ unit vector along radius from the center of Earth towards spacecraft.

If we define a plane (zx) perpendicular to the orbit-plane and containing the target (z) (see Fig. 3),

$$\hat{r}=\begin{bmatrix} c_\psi s_\beta c_{\theta o} & + & s_\psi s_{\theta o} \\ -s_\psi s_\beta c_{\theta o} & + & c_\psi s_{\theta o} \\ c_\beta c_{\theta o} & & \end{bmatrix} \tag{5}$$

where ψ, θo are measured from the zx-plane, and θo measures the orbital position. The gravity-gradient torque about the x- and y-axes is

$$T_{gg}=\begin{bmatrix} T_{ggx}/I_x \\ T_{ggy}/I_y \end{bmatrix}=3n^2\begin{bmatrix} Kc_\beta c_{\theta o}\,(s_{\theta o}c_\psi-s_\beta c_{\theta o}s\psi) \\ -Kc_\beta c_{\theta o}\,(s_{\theta o}s_\psi+s_\beta c_{\theta o}c_\psi) \end{bmatrix} \tag{6}$$

Because of uncertainty in the spacecraft inertia parameters we get an additional gravity-gradient torque component

$$\delta\bar{T}_{gg}=3n^2\begin{bmatrix} \delta K_x c_\beta c_{\theta o}(s_{\theta o}c_\psi-s_\beta c_{\theta o}s_\psi) \\ -\delta K_y c_\beta c_{\theta o}(s_{\theta o}s_\psi+s_\beta c_{\theta o}c_\psi) \end{bmatrix} \tag{7}$$

The nominal gg-torque component about the z-axis is $\bar{T}_{ggz}=1/2K_z[s_\beta s_{2io}c_{2\psi}+(s_{\theta o}{}^2-s_\beta{}^2c_{\theta o}{}^2)s_{2\psi}]$

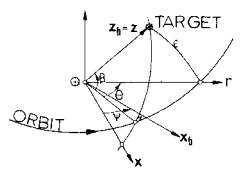
The small periodic torque $\bar{T}_{ggz}$ will not change the spinrate significantly and can be neglected.

Geomagnetic Torque

The geomagnetic torque on the spacecraft is the crossproduct of an effective dipole moment M_s of the spacecraft with the Earth-magnetic field-strength $H_\oplus$

$$T_M=M_s\times H_\oplus \tag{8}$$

Fig. 3 Position and orientation angles.



$\boldsymbol{H}_{\oplus}$ is approximately fixed in space during one spin-revolution. Thus in body axes $\boldsymbol{H}_{\oplus}$ has a constant component along z_b and sinusoidal components along x_b, y_b.

The spacecraft's dipole-moment is fixed to the spacecraft but not necessarily parallel to z_b. Hence $(T_M)_x$, $(T_M)_y$ contain in general both constant and sinusoidal components.

$$\begin{bmatrix} T_{Mx} \\ T_{My} \end{bmatrix} = \begin{bmatrix} a_x + b_x s_\psi - c_x c_\psi \\ a_y + b_y s_\psi + c_y c_\psi \end{bmatrix} \tag{9}$$

Because of ongoing electrical activity on the spacecraft, the parameters $(a,b,c)_{x,y}$ of the geomagnetic torque are not known and have to be estimated.

Radiation-Pressure and Aerodynamic Torque

The radiation-pressure and aerodynamic forces on a surface element of the spacecraft is the change of impulses of photons and molecules respectively. Because of uncertainty in the direction of reflected photons we have a 10 p.c. uncertainty in the effective ratiation pressure torque (RPT). The aerodynamic torque (AT) is only 25 p.c. of the RPT but is predictable with about the same uncertainty as the RPT. Since the center of pressure fluctuates during spin rotation the torque components T_x, T_y have constant and fluctuating parts, in which the first harmonic fluctuation is probably the largest. RPT and AT can thus be included in Eq. (9) for the relatively large magnetic torques.

Torque Due to Misalignment of the Telescope Axis

The equation of motion for a rigid body is described by

$$\overset{I}{H} = I \cdot \overset{B}{\omega}{}^{B-I} + \omega^{B-I} \times (I \cdot \omega^{B-I})$$

For a perfectly known mass distribution the motion is fully defined by Eq. (1). But for uncertainty in the mass distribution we don't know the principal axis of inertia and have to look at the off-diagonal terms of the inertia tensor. For ω_x, $\omega_y \ll \omega_z$ and $\dot{\omega}_{x,y,x} \ll \omega_z{}^2$ we get

$$\begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} = \begin{bmatrix} (I_{xy}/I_x)\omega_z & -K_x\omega_z \\ K_y\omega_z & (I_{xy}/I_y)\omega_z \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \omega_y \end{bmatrix} + \begin{bmatrix} (I_{zy}/I_x)\omega_z{}^2 + T_x/I_x \\ (I_{xz}/I_y)\omega_z{}^2 + T_y/I_y \end{bmatrix} \tag{10}$$

The off-diagonal terms I_{yz}, I_{xz} are due to telescope misalignments and can be observed by the motion of the target in the viewfield of the telescope. Since they are multiplied by the square of the spinrate they can produce a disturbance torque even larger than the nominal gravity-gradient torque. We therefore have to equip the spacecraft with a mass-trim system.[5,6]

After observing the target for some time we can estimate any serious misalignment of the telescope axis and reduce the corresponding disturbance torque by the mass-trim system to a value comparable with the other inaccurately known "constant" disturbance torques a_x, a_y. The terms $I_{xy}\omega_z\omega_x, I_{xy}\omega_z\omega_y$ in Eq. (10) are then negligible. The mass-trim system will not be involved in the further "fine" pointing control of the telescope axis.

Dynamics of the Controlled System

The fine control will consist of two parts; a torque to counteract the nominal gg-torque and an additional torque to offset the residual disturbance torques. The disturbance control torques u_x, u_y may then be expressed as:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = -3n^2 \begin{bmatrix} K_x c_\beta c_{\theta 0}(s_{\theta 0} c_\psi - s_\beta c_{\theta 0} s_\psi) \\ -K_y c_\beta c_{\theta 0}(s_{\theta 0} + s_\beta c_{\theta 0} c_\psi) \end{bmatrix} + \begin{bmatrix} \delta u_x \\ \delta u_y \end{bmatrix} \tag{11}$$

Treating the higher-order terms as random disturbances and introducing the dimensionless time $\psi = \bar{\omega}_z t$, the equations of motion simplify to

$$\begin{bmatrix} \Phi' \\ \Theta' \\ \Phi'' \\ \Theta'' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_x & 0 & 0 & 1-K_x \\ 0 & -K_y & K_y-1 & 0 \end{bmatrix} \begin{bmatrix} \Phi \\ \Theta \\ \Phi' \\ \Theta' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_x + b_x c_\psi + c_x s\psi \\ a_y + b_y c_\psi + c_y s_\psi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \delta u_x \\ \delta u_y \end{bmatrix} + \Gamma \boldsymbol{w} \tag{12}$$

where the primid denotes $\mathrm{d}/\mathrm{d}\psi$. $(a,b,c)_{x,y}$ now include $\boldsymbol{T}_M$, RPT, AT; $(b,c)_{x,y}$ include $\delta \boldsymbol{T}_{gg}$ from Eq. (7) and $(a)_{x,y}$ include fluctuations in control torques and disturbance torques due to misalignment errors left uncorrected by the mass-trim system.

Modeling of the Uncertain Parameters as First-Order Gauss- Markov Processes

We model the uncertain parameters $(a,b,c)_{x,y}$ as Gauss-Markov random processes with time constant τ. The time constant τ we choose somewhat larger than the spin period but much smaller than the orbital period: $a_x' = -(1/\tau)a_x + w, \ldots$ Our system equation becomes then

$$\begin{bmatrix} \phi' \\ \theta' \\ \phi'' \\ \theta'' \\ a'_x \\ a'_y \\ b'_x \\ b'_y \\ c'_x \\ c'_y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & & & & & & \\ 0 & 0 & 0 & 1 & & & 0 & & & \\ -K_x & 0 & 0 & 1-K_x & 1 & 0 & c_\psi & s_\psi & 0 & 0 \\ 0 & -K_y & K_y-1 & 0 & 0 & 1 & 0 & 0 & c_\psi & s_\psi \\ & & & & -(1/\tau\omega_z) & & & & & \\ & & & & & \cdot & & & & \\ & & 0 & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & (1/\tau\omega_\tau) \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \phi' \\ \theta' \\ a_x \\ a_y \\ b_x \\ b_y \\ c_x \\ c_y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \delta u_x \\ \delta u_y \\ \\ \\ 0 \\ \\ \\ \\ \end{bmatrix} \Gamma\omega \tag{13}$$

Optimal Control Law

Since the random disturbances do not contribute to the optimal control law,[7] Eq. (13) can be written

$$\begin{bmatrix} x_1' \\ \\ x_2' \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12}(\psi) \\ \\ 0 & \bar{\omega} I \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} + \begin{bmatrix} G \\ \\ 0 \end{bmatrix} \delta u \qquad (14)$$

where

$$x_1 = \begin{bmatrix} \phi \\ \theta \\ \phi' \\ \theta' \end{bmatrix} \quad x_2 = \begin{bmatrix} a_x \\ a_y \\ b_x \\ b_y \\ c_x \\ c_y \end{bmatrix} \quad \bar{\omega} = (1/\tau\bar{\omega}_z)$$

$$F_{11} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -K_x & -K_y & K_y-1 & 0 \\ 0 & -K_y & K_y-1 & 0 \end{bmatrix}$$

$$F_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & c_\psi & s_\psi & 0 & 0 \\ 0 & 1 & 0 & 0 & c_\psi & a_\psi \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_{12}(\psi+2\pi) = F_{12}(\psi)$$

The control criterion adopted is quadratic in x_1 and δu

$$J = \int_{t_0}^{t_f \to \infty} \tfrac{1}{2}(x_1^T A\, x_1 + \delta u^T \delta u)\,d\psi \qquad (15)$$

The optimal control law is given by

$$\delta u = -G^T[S_{11} S_{12}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -C(\psi)x \qquad (16)$$

where S_{11}, S_{12} is the backward limit of the solution of

$$(S_{11})' = -S_{11}F_{11} - F_{11}^T S_{11} + S_{11}GG^T S_{11} - A \qquad (17)$$

$$(S_{12})' = [S_{11}GG^T - F_{11}^T + \omega I]\, S_{12} - S_{11}F_{12}(\psi) \qquad (18)$$

$$(S_{22})' = 2\omega S_{22} + S_{12}^T GG^T S_{11} - S_{12}^T F_{12}(\psi) - F_{12}^T(\psi) S_{12} \qquad (19)$$

The limiting solution of Eqs. (17-19) can be found by method of Ref. 3 but more easily as follows: Eq. (17) is a MRE for a constant coefficient system, whose steady-state value can be found by the method of eigenvector decomposition.[8,10] Equations (18) and (19) are linear matrix differential equations, with constant coefficients and periodic forcing terms, whose solutions under backward integration approach periodic behavior (with additive constants), obtainable by elementary means. Typical control gains are shown in Figs. 4 and 5, where C_{xi}, C_{yi} are the control gains for δu_x, δu_y respectively.

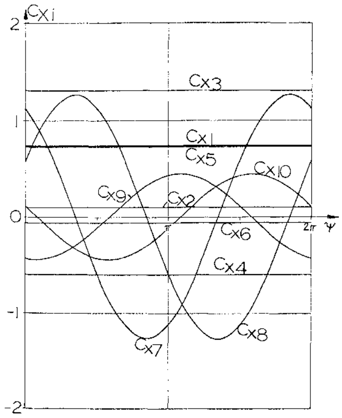


Fig. 4 Control gains C_{xi}.

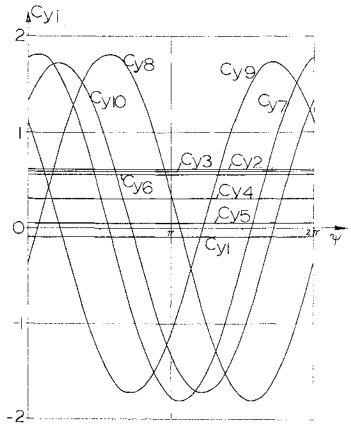


Fig. 5 Control gains C_{yi}.

Estimator

The measurements of the spacecraft's orientation are assumed to be of two types: a) The deviation, measured continuously in body axes, of the telescope axis from the desired direction

$$\zeta_1 = H_1 x + v_1; \quad v_1 = N(O, R_1) \qquad (20)$$

b) the deviation of the telescope axis towards or away from certain bright stars, measured as the star in question crosses a slit on the outside of the spacecraft.[9] The SPARS (Space Precision Attitude Reference System) concept of attitude determination was developed in the early 1960s as a reference system for precision pointing at targets on Earth. For interplanetary missions it has been used successfully for the Pioneer series of spacecraft. To find the attitude of the spacecraft, the time-difference between passages of a bright star past a slit aligned with the spin axis and a slit at an angle α is measured (see Fig. 6). This time difference is proportional to the tilt of the z_b – axis toward the star.

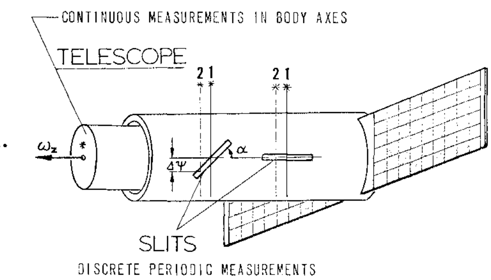


Fig. 6 Attitude reference system.

$$\zeta_{2i}=H_2(\psi_i)x(\psi_i)+\nu_{2i},\ i=1,\ldots,N,\ldots \quad (21)$$
$$E[\nu_i\nu_j]=R_i\delta_{ij}$$

where $H_2(\psi)$ is periodic, and N is the number of discrete observations during one spin-rotation. Because of the constant spin-rate of the spacecraft this measurement is repeated periodically with spin-period.

The Kalman-Bucy estimate of the controlled system is

$$\mathrm{d}\hat{x}/\mathrm{d}\psi=[F-GC]\,\hat{x}+P_pH_1^TR_1^{-1}(\zeta_1-H_1\hat{x})$$
$$+\sum_i P_p(\psi_i^-)H_2^TR_2^{-1}(\zeta_2-H_2\hat{x}(\psi_i^-))\delta(\psi-\psi_i) \quad (22)$$

where $P_p(\psi)$ is the periodic limit of

$$(\mathrm{d}/\mathrm{d}\psi)P=FP+PF^T+\Gamma Q\Gamma^T-PH_1^TR_1^{-1}H_1P$$
$$-\sum_i P(\psi_i^-)H_2^T[H_2P(\psi_i^-)H_2^T+R_2]^{-1}\cdot H_2P(\psi_i^-)\delta(\psi-\psi_i) \quad (23)$$

As shown in Appendix A by an extension of the procedures of T. Nishimura[2] and Breakwell, Kamel, Ratner.[3]

$$P_p(\psi)=\phi_{11}(\psi,0)T_{11}+\phi_{12}(\psi,0)T_{21}][\phi_{21}(\psi,0)T_{11}$$
$$+\phi_{22}(\psi,0)T_{21}]^{-1} \quad (24)$$

where $\phi(\psi,0)$ is the transition matrix associated with the homogeneous part of Eq. (A9) of Appendix A.

The covariance $\hat{X}$ of the estimate $\hat{x}$ of x is defined by $\hat{X}=E[(\hat{x}-E\hat{x})(\hat{x}-E\hat{x})^T]$, which satisfies

$$(\mathrm{d}/\mathrm{d}\psi)\hat{X}=[F-GC(\psi)]\,\hat{X}+\hat{X}[F-GC(\psi)]^T$$
$$+P_pH_1^TR_1^{-1}H_1P_p+\sum_i P_p(\psi_i^+)H_2^TR_2^{-1}[R_2+$$
$$H_2P_p(\psi_i^-)H_2^T]R_2^{-1}H_2P_p(\psi_i^+)\delta(\psi-\psi_i) \quad (25)$$

$\hat{X}$ approaches a periodic $\hat{X}_p(\psi)$

$$\hat{X}_p(\psi)=\phi^*(\psi,0)\hat{X}_p(0)\phi^{*T}(x,0)$$
$$+\int_0^{\psi}\phi^*(\psi,\tau)g(\tau)\phi^{*T}(\psi,\tau)\mathrm{d}\tau \quad (26)$$

where
$$g(\tau)=P_pH_1^TR_1^{-1}H_1P_p+\sum_i P_p(\psi_i^+)H_2^TR_2^{-1}[R_2$$
$$+H_2P_p(\psi_i^-)H_2^T]R_2^{-1}H_2P(\psi_i)\delta(\psi-\psi_i)$$

$\phi^*(.,.)$ is the transition matrix associated with the homogeneous equation $(\mathrm{d}/\mathrm{d}\psi)\hat{x}=[F-GC(\psi)]\hat{x}$, and $\hat{X}_p(0)$ is chosen so that $\hat{X}_p(2\pi)=\hat{X}_p(0)$.

Hence we get the initial condition for Eq. (26) by solving the linear equation.

$$\hat{X}_p(0)=\phi^*(2\pi,0)\hat{X}_p(0)\phi^{*T}(2\pi,0)$$
$$+\int_0^{2\pi}\phi^*(2\pi,\tau)g(\tau)\phi^{*T}(2\pi,\tau)\mathrm{d}\tau \quad (27)$$

The covariance $E[x(\psi)x^T(\psi)]$ at normalized times ψ much larger than the control settling time and information settling time is given by the sum of $\hat{X}_p(\psi)$ and $P_p(\psi)$.

We get the average deviation from the desired pointing direction from

$$\mathrm{rms}\,|r|=\left\{1/2\pi\int_0^{2\pi}\mathrm{tr}[0,I,0]E[x(\psi)x^T(\psi)]\,\mathrm{d}\psi\right\}^{1/2} \quad (28)$$

where $r=[\phi\ \theta]^T$, and the average control torque is

$$\mathrm{rms}\,|u|=\left\{1/2\pi\int_0^{2\pi}\mathrm{tr}[u_p(\psi)u_p(\psi)^T+G^TSX_pSG]\mathrm{d}\psi\right\}^{1/2} \quad (29)$$

where $u_p(\psi)$ is the control torque in Eq. (11).

Conclusions

General conclusions from this study of the LSST are: 1) The various disturbance torques can be expressed by means of 6 additional states obeying first-order Gauss-Markov laws, the augmented linear dynamic system having coefficients which are periodic with spin period. 2) The optimal periodic control gains can be easily calculated by seperating the uncontrollable states from the controllable states. 3) The method of spectral factorization provides periodic gains for the state estimator not only from continuous but also from discrete, periodically repeated, observations. Further effort is necessary to get numerical results for rms-values of the pointing accuracy to compare the LSST with the LST.

Appendix A: Optimal Filter for Periodic Systems with Discrete Periodically Repeated Measurements

To find the optimal filter we have to minimize

$$\min_{w(\tau)}\left\{\tfrac{1}{2}\|x(0)\|^2_{P(0)^{-1}}+\int_0^{\psi}L(x,w,\tau)\mathrm{d}\tau\right\}\triangleq J^{(o)}(x,\psi)$$
$$x(\psi)=x \quad (\mathrm{A1})$$

subject to

$$(\mathrm{d}x/\mathrm{d}\psi)=F(\psi)x+Gu+\Gamma w,\quad w=N(0,Q)$$

where the piecewise continuous function $L(x,w,\psi)$ is denoted by

$$L(x,w,\psi)\triangleq\tfrac{1}{2}\{\|\zeta_1-H_1x\|^2_{R_1^{-1}}$$
$$+\|w\|^2_{Q^{-1}}+\sum_i\|\zeta_2-H_2(\psi)x\|^2_{R_{2i}^{-1}}\delta(\psi-\psi)\} \quad (\mathrm{A2})$$

with ζ_1 = continuous measurements, and ζ_2 = discrete periodically repeated measurements.

To find the optimal estimate of the state we have to satisfy the Hamilton-Jacobi-Bellman equation

$$-J^o_\psi+\underset{w}{\mathrm{Min}}(L-J_xx')=0 \quad (\mathrm{A3})$$

so that

$$w=Q^{-1}\Gamma^TJ_x^T \quad (\mathrm{A4})$$

Assuming a quadratic form as a solution of Eq. (A3),

$$J^o(x,\psi)=\tfrac{1}{2}\{\|x-\hat{x}\|^2_{P^{-1}(\psi)}\} \quad (\mathrm{A5})$$

we deduce by minimization the Riccati-type matrix differential-difference equation

$$\mathrm{d}/\mathrm{d}\psi(P^{-1})=-P^{-1}F-F^TP^{-1}-P^{-1}\Gamma Q\Gamma^TP^{-1}$$
$$+H_1^TR_1^{-1}H_1+\sum_i H_2^TR_2^{-1}H_2\delta(\psi-\psi_i) \quad (\mathrm{A6})$$

and the Kalman-Bucy filter

$$\mathrm{d}\hat{x}/\mathrm{d}\psi=F\hat{x}+Gu+PH_1^TR_1^{-1}[\zeta_1-H_1\hat{x}]$$
$$+\sum_i P^+H_2^TR_2^{-1}[\zeta_{2i}-H_2\hat{x}^-]\delta(\psi-\psi_i),\hat{x}(0)=0 \quad (\mathrm{A7})$$

Defining $\mu\triangleq J_x^T=P^{-1}(x-\hat{x})$ we get the adjoint equation for the estimator

$$\mathrm{d}\mu/\mathrm{d}\psi=-F^T\mu-H_1^TR_1^{-1}(\zeta_1-H_1x)$$
$$-\sum_i H_2^TR_{2i}^{-1}[\zeta_2-H_2x]\delta(\psi-\psi_i) \quad (\mathrm{A8})$$

Adjoining the μ-equation (A8) to the x-equation (A1) we get, using Eq. (A4), the joint system equation.

$$\frac{d}{d\psi}\begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} F & \Gamma Q \Gamma^T \\ H_1^T R_1^{-1} H_1 + \sum H_2^T R_2^{-1} H_2 \delta(\psi - \psi_i) & -F^T \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} + \begin{bmatrix} G\mu \\ H_1^T R_1^{-1} \zeta_1 - \sum_i H_2^T R_2^{-1} \zeta_{2i} \delta(\psi - \psi_i) \end{bmatrix} \quad \text{(A9)}$$

Integrating Eq. (A9) we get $x(\psi)$, $\mu(\psi)$

$$\begin{bmatrix} x(\psi) \\ \mu(\psi) \end{bmatrix} = \Phi(\psi,0) \begin{bmatrix} x(0) \\ \mu(0) \end{bmatrix} + \int_o^\psi \Phi(\psi,\tau)$$

$$\times \begin{bmatrix} G u \\ -H_1^T R_1^{-1} \zeta_1 - \sum_i H_2^T R_2^{-1} \zeta_{2i} \delta(\tau - \psi_i) \end{bmatrix} d\tau \quad \text{(A10)}$$

where $\Phi(.,.)$ is the transition matrix associated with the homogeneous part of Eq. (A9).

Since the homogeneous part of Eq. (A9) is of "Hamiltonian" form, its transition matrices $\Phi(.,.)$ are "symplectic." Hence $\Phi(2\pi,0)$ has eigenvalues in pairs $(v_i, 1/v_i)$. Suppose that $\Phi(2\pi,0)$ has 10 distinct eigenvalues v_i with $|v_i|>1$. Let $[T_{11} T_{21}]^T$ have columns which are eigenvectors of $\Phi(2\pi,0)$ corresponding to the 10 eigenvalues v_i. Then, if $u(\psi) = \zeta(\psi) = 0$,

$$\hat{x}(\psi) = 0, \; x = P(\psi)\mu,$$

and

$$\begin{bmatrix} x(2M\pi) \\ \mu(2M\pi) \end{bmatrix} = \Phi(2\pi,0)^M \begin{bmatrix} x(0) \\ \mu(0) \end{bmatrix} \quad \text{(A11)}$$

As $M \to \infty$ $x(2M\pi) \to T_{11} T_{21}^{-1} \mu(2M\pi)$. But $x(\psi) = P\mu(\psi)$,hence $P(2M\pi) \to T_{11} T_{21}^{-1}$ as $M\infty$. By forward integration

$$\begin{bmatrix} x(\psi + 2M\pi) \\ \mu(\psi + 2M\pi) \end{bmatrix} = \Phi(\psi,0) \begin{bmatrix} I \\ P^{-1}(0) \end{bmatrix}$$

$$x(2M\pi) = \begin{bmatrix} \Phi_{11} + \Phi_{12} P^{-1}(0) \\ \Phi_{21} + \Phi_{22} P^{-1}(0) \end{bmatrix} x(2M\pi)$$

Thus the limit of the error covariance matrix is

$$P_p(\psi) = [\Phi_{11}(\psi,0) T_{11} + \Phi_{12}(\psi,0) T_{21}]$$

$$\times [\Phi_{21}(\psi,0) T_{11} + \Phi_{22}(\psi,0) T_{21}]^{-1}$$

as

$$M \mapsto \infty, \; 0 \le \psi \ge 2\pi$$

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